The Reverse Ultra Log-Concavity of the Boros-Moll Polynomials

William Y.C. Chen[1], Cindy C.Y. Gu[2]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China
[1]chen@nankai.edu.cn, [2]guchunyan@cfc.nankai.edu.cn

Abstract. We prove the reverse ultra log-concavity of the Boros-Moll polynomials. We further establish an inequality which implies the log-concavity of the sequence $\{i!d_i(m)\}$ for any $m \geq 2$, where $d_i(m)$ are the coefficients of the Boros-Moll polynomials $P_m(a)$. This inequality also leads to the fact that in the asymptotic sense, the Boros-Moll sequences are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We propose two conjectures on the log-concavity and reverse ultra log-concavity of the sequence $\{d_{i-1}(m)d_{i+1}(m)/d_i(m)^2\}$ for $m \geq 2$.

Keywords: log-concavity, reverse ultra log-concavity, Boros-Moll polynomials.

AMS Classification: 05A20; 33F10

1 Introduction

This paper is concerned with the reverse ultra log-concavity of the Boros-Moll polynomials. A sequence $\{a_k\}_{k\geq 0}$ of real numbers is said to be log-concave if $a_k^2 \geq a_{k+1}a_{k-1}$ holds for all $k \geq 1$. A polynomial is said to be log-concave if the sequence of its coefficients is log-concave, see Brenti [7] and Stanley [10]. Furthermore, a sequence $\{a_k\}_{0\leq k\leq n}$ is called ultra log-concave if $\left\{a_k/\binom{n}{k}\right\}$ is log-concave, see Liggett [9]. This condition can be restated as

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \tag{1.1}$$

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. As noticed by Liggett [9], if a sequence $\{a_k\}_{0\leq k\leq n}$ is ultra log-concave, then the sequence $\{k!a_k\}_{0\leq k\leq n}$ is log-concave.

A sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (1.1), that is,

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0. \tag{1.2}$$

For example, it is easy to verify that for $n \geq 2$, the Bessel polynomial [11]

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^k$$

is log-concave and reverse ultra log-concave.

The Boros and Moll polynomials, denoted $P_m(a)$, arise in the following evaluation of a quartic integral

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a),$$

where

$$P_m(a) = 2^{-2m}\sum_k 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k, \tag{1.3}$$

see, [1, 2, 3, 5]. Write

$$P_m(a) = \sum_{i=0}^m d_i(m)a^i.$$

The sequence $\{d_i(m)\}_{0\le i\le m}$ is called a Boros-Moll sequence. The expression (1.3) gives the following formula for the coefficients $d_i(m)$,

$$d_i(m) = 2^{-2m}\sum_{k=0}^m 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{i}.$$

Clearly, the coefficients $d_i(m)$ are positive. Moll conjectured that the sequence $\{d_i(m)\}_i$ is log-concave for $m\ge 2$, that is, $d_i(m)^2 \ge d_{i-1}(m)d_{i+1}(m)$ $(1\le i\le m-1)$. This conjecture has been proved by Kauers and Paule [8].

Despite the log-concavity of $\{d_i(m)\}$, we find that the inverse ultra log-concavity holds.

Theorem 1.1 *For $m\ge 2$ and $1\le i\le m-1$, we have*

$$\left(\frac{d_{i-1}(m)}{\binom{m}{i-1}}\right)\cdot\left(\frac{d_{i+1}(m)}{\binom{m}{i+1}}\right) > \left(\frac{d_i(m)}{\binom{m}{i}}\right)^2, \tag{1.4}$$

or, equivalently,

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i}. \tag{1.5}$$

On the other hand, it can be shown that the coefficients $d_i(m)$ satisfy an inequality stronger than the log-concavity. To be more specific, we will give a lower bound of $d_i(m)^2/\left(d_{i-1}(m)d_{i+1}(m)\right)$, which is very close to the above upper bound in (1.5).

Theorem 1.2 *For $m\ge 2$ and $1\le i\le m-1$, we have*

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)}. \tag{1.6}$$

This paper is organized as follows. We establish an upper bound of $d_i(m+1)/d_i(m)$ in Section 2, which leads to the reverse ultra log-concavity of $\{d_i(m)\}$. In Section 4 we give the proof of Theorem 1.2. We conclude this paper with two conjectures concerning the log-concavity and the reverse ultra log-concavity of the sequence $\{d_{i-1}(m)d_{i+1}(m)/d_i^2(m)\}$ for $m \geq 2$.

2 An Upper Bound for $d_i(m+1)/d_i(m)$

In this section, we establish an upper bound for the ratio $d_i(m+1)/d_i(m)$ that will lead to the reverse ultra log-concavity of the sequence of $\{d_i(m)\}$. For $m \geq 1$ and $0 \leq i \leq m$, set

$$T(m,i) = \frac{4m^2+7m+3+i\sqrt{4m+4i^2+1}-2i^2}{2(m-i+1)(m+1)}. \tag{2.1}$$

Theorem 2.1 *For all $m \geq 2$, $1 \leq i \leq m-1$, we have*

$$\frac{d_i(m+1)}{d_i(m)} < T(m,i), \tag{2.2}$$

and for $m \geq 1$, we have

$$\frac{d_0(m+1)}{d_0(m)} = T(m,0), \qquad \frac{d_m(m+1)}{d_m(m)} = T(m,m). \tag{2.3}$$

The following lemma will be needed in the proof of Theorem 2.1.

Lemma 2.2 *For $m \geq 2$ and $1 \leq i \leq m-1$,*

$$T(m,i) < F(m,i), \tag{2.4}$$

where

$$F(m,i) = \frac{(m+i+1)(4m+3)(4m+5)}{2(m+1)(4m^2-2i^2+9m+5-i\sqrt{4m+4i^2+5})}.$$

Proof. Let $A = \sqrt{4m+4i^2+1}$ and $B = \sqrt{4m+4i^2+5}$. It is easy to check that

$$F(m,i) - T(m,i) = \frac{i(X-Y)}{2(m+1)(m-i+1)(4m^2+9m+5-2i^2-iB)}, \tag{2.5}$$

where

$$\begin{aligned} X =& (i-4i^3) + iAB \\ Y =& (5+4m^2+9m-2i^2)A - (3+4m^2+7m-2i^2)B. \end{aligned}$$

Since $(4m^2+9m+5-2i^2)^2-(iB)^2=(4m+5)^2(m+i+1)(m-i+1)>0$, it remains to show that the numerator of (2.5) is also positive. We claim that $X>0$ and $X^2>Y^2$.

Since $m>i$, we have $A>2i+1$ and $B>2i+1$. Moreover, since $i\geq 1$, we find that

$$X=(i-4i^3)+iAB\geq i-4i^3+i(2i+1)^2=4i^2+2i>0.$$

It is routine to check $X^2-Y^2=G(m,i)-H(m,i)$, where

$$\begin{aligned}
G(m,i) &= (32m^4-32m^2i^2+128m^3-64mi^2+190m^2-30i^2+124m+30)AB,\\
H(m,i) &= 128m^5+608m^4+1128m^3+1014m^2+436m+128m^4i^2+384m^3i^2\\
&\quad +408m^2i^2-128m^2i^4+200mi^2-256mi^4-120i^4+50i^2+70.
\end{aligned}$$

Since $i<m$, it is easily seen that $G(m,i)>0$ and $H(m,i)>0$. To prove $G(m,i)>H(m,i)$, it suffices to show that $G(m,i)^2>H(m,i)^2$. In fact, for $1\leq i\leq m-1$,

$$G(m,i)^2-H(m,i)^2=16(4m+5)^2(16mi^2+12i^2-1)(m+i+1)^2(m-i+1)^2>0.$$

This yields $X^2>Y^2$. Since $X>0$, we see that $X>Y$, and hence (2.4) holds for $1\leq i\leq m-1$. ∎

Proof of Theorem 2.1. It is easy to check (2.3). To prove (2.2), we proceed by induction on m. For $m=2$ and $i=1$, we have $d_1(3)/d_1(2)=43/15<T(2,1)=(31+\sqrt{13})/12$. We now assume that (2.2) is true for m, that is,

$$d_i(m+1)<T(m,i)d_i(m),\quad 1\leq i\leq m-1. \tag{2.6}$$

It will be shown that

$$d_i(m+2)<T(m+1,i)d_i(m+1),\quad 1\leq i\leq m-1. \tag{2.7}$$

Using the recurrence (3.3), we may write (2.7) in the following form

$$\begin{aligned}
\frac{-4i^2+8m^2+24m+19}{2(m-i+2)(m+2)}d_i(m+1)-\frac{(m+i+1)(4m+3)(4m+5)}{4(m+1)(m+2)(m-i+2)}d_i(m)&\\
<T(m+1,i)d_i(m+1).&
\end{aligned} \tag{2.8}$$

Since $m>i$, we have $4m+4i^2+5<12m+4m^2+9$. It follows that

$$\begin{aligned}
R(m,i)&=\frac{-4i^2+8m^2+24m+19}{2(m-i+2)(m+2)}-T(m+1,i)\\
&=\frac{4m^2+9m+5-2i^2-i\sqrt{4m+4i^2+5}}{2(m-i+2)(m+2)}\\
&\geq\frac{4m^2+9m+5-2i^2-i(2m+3)}{2(m-i+2)(m+2)}>0.
\end{aligned}$$

Therefore, (2.8) is equivalent to the following inequality

$$\frac{d_i(m+1)}{d_i(m)} < F(m,i), \tag{2.9}$$

which is a consequence of (2.6) and Lemma 2.2.

It remains to consider the case $i = m$. We aim to show that

$$\frac{d_m(m+2)}{d_m(m+1)} < T(m+1,m). \tag{2.10}$$

By easy computation, we find that

$$\frac{d_m(m+2)}{d_m(m+1)} = \frac{(m+1)(4m^2+18m+21)}{2(2m+3)(m+2)},$$

$$T(m+1,m) = \frac{2m^2+15m+14+m\sqrt{4m^2+4m+5}}{4(m+2)}.$$

Thus (2.10) can be rewritten as

$$(2m^2+3m)\sqrt{4m^2+4m+5} > 4m^3+8m^2+5m. \tag{2.11}$$

Denote by U and V the left hand side and the right hand side of (2.11), respectively. Then, $U^2 - V^2 = 4m^2(4m+5) > 0$, and so (2.10) is verified. This completes the proof. ∎

3 The Reverse Ultra Log-concavity

In this section, we give the proof of Theorem 1.1. Our approach can be described as follows. Let $f(x) = ax^2 + bx + c$ be a quadratic function with $a > 0$. Suppose that the equation $f(x) = 0$ has two distinct real zeros x_1 and x_2, where $x_1 < x_2$. Then $f(x) > 0$ if $x > x_2$ or $x < x_1$ and $f(x) < 0$ if $x_1 < x < x_2$. The key step is to transform the inequality (1.5), that is,

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i},$$

into a quadratic inequality in the ratio $d_i(m+1)/d_i(m)$.

We will need the following recurrence relations for the coefficients $d_i(m)$. For $m \geq 1$ and $0 \leq i \leq m+1$,

$$2(m+1)d_i(m+1) = 2(m+i)d_{i-1}(m) + (4m+2i+3)d_i(m), \tag{3.1}$$

$$\begin{aligned} 2(m+1)(m+1-i)d_i(m+1) &= (4m-2i+3)(m+i+1)d_i(m) \\ &\quad - 2i(i+1)d_{i+1}(m), \end{aligned} \tag{3.2}$$

$$\begin{aligned} 4(m+2-i)(m+1)(m+2)d_i(m+2) &= 2(m+1)(-4i^2+8m^2+24m+19)d_i(m+1) \\ &\quad - (m+i+1)(4m+3)(4m+5)d_i(m). \end{aligned} \tag{3.3}$$

These recurrences are derived by Kauers and Paule [8]. The relation (3.3) is also derived independently by Moll [6]. Based on these recurrence relations, Kauers and Paule [8] derived the following lower bound of $d_i(m+1)/d_i(m)$ in their proof of the log-concavity of Boros-Moll polynomials

$$\frac{d_i(m+1)}{d_i(m)} \geq Q(m,i), \quad 0 \leq i \leq m, \tag{3.4}$$

where

$$Q(m,i) = \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}. \tag{3.5}$$

Note that Chen and Xia [4] have shown that the above inequality (3.4) becomes strict for $m \geq 2$ and $1 \leq i \leq m-1$, that is,

$$\frac{d_i(m+1)}{d_i(m)} > Q(m,i). \tag{3.6}$$

Now we are ready to prove the reverse ultra log-concavity of $\{d_i(m)\}$.

Proof of Theorem 1.1. Applying (3.1) and (3.2), we may reformulate (1.5) in the following form

$$\begin{aligned} &4(m-i+1)^2(m+1)^2\left(\frac{d_i(m+1)}{d_i(m)}\right)^2 \\ &-4(m-i+1)(m+1)(4m^2-2i^2+7m+3)\left(\frac{d_i(m+1)}{d_i(m)}\right) \\ &-(32mi^2-56m^3-73m^2-42m+13i^2-9-16m^4+16i^2m^2)<0. \end{aligned} \tag{3.7}$$

For $1 \leq i \leq m-1$, the discriminant of the above quadratic function in $d_i(m+1)/d_i(m)$ equals

$$\triangle = 16i^2(m+1)^2(4i^2+4m+1)(m-i+1)^2 > 0.$$

We see that the quadratic function on the left hand side of (3.7) has two real roots

$$x_1 = \frac{4m^2-2i^2+7m+3-i\sqrt{4m+4i^2+1}}{2(m-i+1)(m+1)},$$

$$x_2 = \frac{4m^2-2i^2+7m+3+i\sqrt{4m+4i^2+1}}{2(m-i+1)(m+1)}.$$

Clearly, $Q(m,i) > x_1$. In view of (3.4), we deduce that $d_i(m+1)/d_i(m) \geq Q(m,i) > x_1$. Observe that x_2 coincides with the upper bound $T(m,i)$ in Theorem 2.1. Thus we have $d_i(m+1)/d_i(m) < x_2$. So we have shown that for $1 \leq i \leq m-1$,

$$x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2,$$

which implies (3.7). This completes the proof of Theorem 1.1. ∎

4 A Lower Bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$

In this section, we give the proof of Theorem 1.2 on a lower bound of $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$. As will be seen, the lower bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$ is very close to the upper bound (1.5) for the reverse ultra log-concavity. So in the asymptotic sense, we may say that the Boros-Moll polynomials are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We conclude this paper with two conjectures.

Proof of Theorem 1.2. Utilizing the recurrence relations (3.1) and (3.2), the inequality (1.6) can be restated as

$$\begin{aligned}
&4(m+1)^2(m-i+1)^2\left(\frac{d_i(m+1)}{d_i(m)}\right)^2\\
&-4(m-i+1)(m+1)(4m^2+7m-2i^2+3)\frac{d_i(m+1)}{d_i(m)}\\
&+(4m^2+7m+3)(-4i+3+4m)(m+i+1)>0.
\end{aligned}$$

For $1\leq i\leq m-1$, the discriminant of the above quadratic function in $d_i(m+1)/d_i(m)$ equals

$$\delta=16i^2(2i+1)^2(m+1)^2(m-i+1)^2>0. \tag{4.1}$$

Hence the above quadratic function has two real roots,

$$\begin{aligned}
x_1&=\frac{4m^2+7m-4i^2-i+3}{2(m+1)(m-i+1)},\\
x_2&=\frac{4m^2+7m+i+3}{2(m+1)(m-i+1)}.
\end{aligned}$$

As $x_2=Q(m,i)$, it follows from (3.6) that $d_i(m+1)/d_i(m)>x_2$. So we arrive at (1.6). This completes the proof. ∎

Notice that for $1\leq i\leq m-1$,

$$\frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)}>\frac{i+1}{i}.$$

As a consequence of Theorem 1.2 , we obtain the log-concavity of the sequence $\{i!d_i(m)\}$.

Corollary 4.1 *For $m\geq 2$ and $1\leq i\leq m-1$,*

$$\frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)}>\frac{i+1}{i}, \tag{4.2}$$

or equivalently, the sequences $\{i!d_i(m)\}$ is log-concave.

Corollary 4.2 *For $1 \leq i \leq m-1$, let*

$$c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} \quad \textit{and} \quad u_i(m) = \left(1+\frac{1}{i}\right)\left(1+\frac{1}{m-i}\right).$$

Then for any $i \geq 1$,

$$\lim_{m\to\infty} \frac{c_i(m)}{u_i(m)} = 1. \tag{4.3}$$

Proof. By Theorems 1.1 and 1.2, we find that

$$\frac{m+i}{m+i+1} < \frac{c_i(m)}{u_i(m)} < 1,$$

which implies (4.3). ∎

We remark that even when m is small, $c_i(m)$ is quite close to $u_i(m)$ for any $1 \leq i \leq m-1$. Numerical evidence indicates that $c_i(m)/u_i(m)$ is increasing for given m. For example, when $m = 8$, the values of $c_i(m)/u_i(m)$ for $1 \leq i \leq 7$ are given below

0.956593, 0.969751, 0.978293, 0.983956, 0.987811, 0.990507, 0.992445.

We propose the following two conjectures on the log-concavity and reverse ultra log-concavity of the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}$.

Conjecture 4.3 *For $m \geq 2$, the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2\leq i\leq m-2}$ is log-concave.*

Conjecture 4.4 *For $m \geq 2$, the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2\leq i\leq m-2}$ is reverse ultra log-concave.*

Acknowledgments. The authors would like to thank the referee for the valuable comments leading to an improvement of an earlier version. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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